WS #5 - OR in the log-linear model

Math 150, Jo Hardin

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Your Name: ____

Names of people you worked with: ____

Where did you come from before class? Which door did you use to enter Estella (north, south, or west)?

Task: Consider a set-up very similar to logistic regression. The response variable is still binary, and a generalized linear model is still applied. However, a different link function is used. The model is sometimes referred to as the log-linear model.[1]

$$
\begin{aligned}
E(Y|x) &= p(x) \\
\ln(p(x)) &= \beta_0 + \beta_1 x
\end{aligned}
$$

In terms of β_0 and β_1, find:

1. The odds ratio for a one unit increase in x (your answer should depend on x).
2. The relative risk for a one unit increase in x (your answer should not depend on x).

[1] n.b., As described in the notes, this model is often unstable when applied to a binary response variable because there is no way to keep $p(x)$ bounded below 1. The instability often leads to parameter estimates that do not converge / are not estimable via maximum likelihood. The inability of the model to converge is one of the main reasons we use logistic regression instead of log-linear models.

Solution: Recall, odds ratios are defined by two separate calculations of odds – one for each of two groups.

$$p(x) = e^{\beta_0+\beta_1 x}$$

1.

$$\begin{aligned}
odds(x) &= \frac{p(x)}{1-p(x)} = \frac{e^{\beta_0+\beta_1 x}}{1-e^{\beta_0+\beta_1 x}} \\
odds(x+1) &= \frac{p(x+1)}{1-p(x+1)} = \frac{e^{\beta_0+\beta_1 (x+1)}}{1-e^{\beta_0+\beta_1 (x+1)}} \\
OR &= \frac{odds(x+1)}{odds(x)} = \frac{\frac{e^{\beta_0+\beta_1 (x+1)}}{1-e^{\beta_0+\beta_1 (x+1)}}}{\frac{e^{\beta_0+\beta_1 x}}{1-e^{\beta_0+\beta_1 x}}} \\
&= e^{\beta_1} \cdot \frac{1-e^{\beta_0+\beta_1 x}}{1-e^{\beta_0+\beta_1 (x+1)}}
\end{aligned}$$

2.

$$RR = \frac{p(x+1)}{p(x)} = \frac{e^{\beta_0+\beta_1 (x+1)}}{e^{\beta_0+\beta_1 x}} = e^{\beta_1}$$